Multiple zeta values and associators in genus zero and one

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Part I: Multizeta values and associators in genus zero

For each sequence $(k_1, \ldots, k_r)$ of strictly positive integers, $k_1 \geq 2$, the **multiple zeta value** is defined by the convergent series

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

These real numbers have been studied since Euler (1775).

They form a $\mathbb{Q}$-algebra, the *multizeta algebra* $\mathcal{Z}$.

Two multiplications of multizeta values

1. Shuffle multiplication

Straightforward integration shows that

$$\zeta(k_1,\ldots,k_r) = (-1)^r \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n-\epsilon_n} \cdots \frac{dt_2}{t_2-\epsilon_2} \frac{dt_1}{t_1-\epsilon_1}$$

where

$$(\epsilon_1,\ldots,\epsilon_n) = (\underbrace{0,\ldots,0}_{k_1-1},1,\underbrace{0,\ldots,0}_{k_2-1},1,\ldots,\underbrace{0,\ldots,0}_{k_r-1},1).$$

The product of two simplices is a union of simplices, giving an expression for the product of two multizeta values as a sum of multizeta values. This is the **shuffle product**.

Example. We have

$$\zeta(2) = \int_0^1 \int_0^{t_1} \frac{dt_2}{1-t_2} \frac{dt_1}{t_1}$$

$$\zeta(2,2) = \int_0^1 \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \frac{dt_4}{1-t_4} \frac{dt_3}{t_3} \frac{dt_2}{1-t_2} \frac{dt_1}{t_1}$$

$$\zeta(3,1) = \int_0^1 \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \frac{dt_4}{1-t_4} \frac{dt_3}{1-t_3} \frac{dt_2}{t_2} \frac{dt_1}{t_1}$$

and

$$\zeta(2)^2 = 2\,\zeta(2,2) + 4\,\zeta(3,1).$$

2. Stuffle multiplication

The product of two series over ordered indices can be expressed as a sum of series over ordered indices. This is the **stuffle product** of multizeta values.

Examples. We have

$$\zeta(2)^2 = \Big(\sum_{n>0} \frac{1}{n^2}\Big)\Big(\sum_{m>0} \frac{1}{m^2}\Big)$$

$$= \sum_{n>m>0} \frac{1}{n^2 m^2} + \sum_{m>n>0} \frac{1}{n^2 m^2} + \sum_{n=m>0} \frac{1}{n^4}$$

$$= 2\zeta(2,2) + \zeta(4).$$

Using either multiplication shows that the multiple zeta values form a $\mathbb{Q}$-algebra. We use the shuffle multiplication and call this algebra $\mathcal{Z}$.

Transcendence conjecture. *The $\mathbb{Q}$-algebra $\mathcal{Z}$ is graded by the weight.*

Zagier Dimension conjecture. *The dimension of the n-th graded part $\mathcal{Z}_n$ is equal to d_n where $d_n = d_{n-2} + d_{n-3}$ with $d_0 = 1$, $d_1 = 0$, $d_2 = 1$.*

Remark. The algebra of motivic multizetas $\mathcal{MZ}$ surjects onto $\mathcal{Z}$. The dimension conjecture is known for $\mathcal{MZ}$ (Goncharov+Brown) so the d_n provide an upper bound for $dim(\mathcal{Z}_n)$.

Convergent and non-convergent words

A **convergent word** $w \in \mathbb{Q}\langle x, y\rangle$ is a word $w = xvy$.

The reason for this notation is that it gives a bijection

$$\{\text{tuples with } k_1 \geq 2\} \leftrightarrow \{\text{convergent words}\}$$
$$(k_1, \ldots, k_r) \leftrightarrow x^{k_1-1}y \cdots x^{k_r-1}y.$$

As a notation, we use this to write

$$\zeta(k_1, \ldots, k_r) = \zeta(x^{k_1-1}y \cdots x^{k_r-1}y).$$

We extend the definition to $\zeta(w)$ for any word $w = y^a u x^b$ with u convergent:

$$\zeta(w) = \sum_{r=0}^{a} \sum_{s=0}^{b} (-1)^{r+s} \zeta\big(sh(y^r, y^{a-r} u x^{b-s}, x^s)\big).$$

The *depth* of a word w the number of y's and the *weight* is the degree; correspondingly, the *depth* of $\zeta(k_1, \ldots, k_r)$ is r and the *weight* is $k_1 + \cdots + k_r$.

The Drinfel'd associator

Definition. The *Drinfel'd associator* is the power series given by

$$\Phi_{KZ}(x,y) = 1 + \sum_{w \in \mathbb{Q}\langle x,y\rangle} (-1)^{d_w}\zeta(w)w$$

where d_w is the number of y's in the word w. It is a generating series for multizeta values.

Let Φ_{KZ}^r be the depth r part of Φ_{KZ}. Then Φ_{KZ}^r can also be obtained as the iterated integral

$$\Phi_{KZ}^r(x,y) = \int_{0<v_r<\cdots<v_1<1} \Big(\frac{x}{v_1} + \frac{y}{1-v_1}\Big) \cdots \Big(\frac{x}{v_r} + \frac{y}{1-v_r}\Big)\, dv_r \cdots dv_1.$$

It is obtained as monodromy of the KZB equation

$$\frac{d}{dz}G(z) = \Big(\frac{x}{v} + \frac{y}{1-v}\Big)G(z);$$

more specifically $\Phi_{KZ}(x,y) = G_1(z)^{-1}G_0(z)$, where G_0 (resp. G_1) is the solution to the KZ equation that tends to z^x as $z \to 0$ (resp. to $(1-z)^y$ as $z \to 1$).

Associator relations

Theorem. [Drinfel'd] *The Drinfeld associator also satisfies the* associator relations

$$(I) \qquad \Phi_{KZ}(x,y)\Phi_{KZ}(y,x) = 1$$

$$(II) \qquad e^{\pi i x}\Phi_{KZ}(y,x)e^{\pi i y}\Phi_{KZ}(z,y)e^{i\pi z}\Phi_{KZ}(x,z) = 1$$

with $x + y + z = 0$*, and*

(III) *The 5-cycle relation*

$$\Phi_{KZ}(x_{12}, x_{23})\Phi_{KZ}(x_{34}, x_{45})\Phi_{KZ}(x_{51}, x_{12})\Phi_{KZ}(x_{23}, x_{34})\Phi_{KZ}(x_{45}, x_{51}) = 1,$$

where the x_{ij} *generate the pure braid group on 5 strands.*

All these properties of $\Phi_{KZ}(x,y)$ yield algebraic relations between multizeta values.

Conjecture. *The double shuffle relations generate all algebraic relations between multizeta values.*

Much weaker conjecture! *The double shuffle relations generate all algebraic relations between motivic multizeta values.*

Definition. We say that a power series $\Phi(x,y) \in \mathbb{C}\langle\langle x,y\rangle\rangle$ is an *associator* if it is group-like with no linear term and satisfies the associator relations (I), (II), (III) with $i\pi$ replaced by $\mu/2$ for some $\mu \in \mathbb{C}$. Let GRT_μ be the set of associators $\Phi(x,y)$ with a fixed μ. In particular $\Phi_{KZ} \in GRT_{2\pi i}$.

Application of associators

Formality: Every associator with $\mu \neq 0$ gives rise to a *formality isomorphism* between the pro-unipotent group

$$\pi_1^{pro-uni}\big(\mathbb{P}^1 - \{0,1,\infty\}\big) \simeq \langle e^X, e^Y, e^Z | e^X e^Y e^Z = e^{CH(X,Y,Z)} = 1\rangle$$

and the graded version $\exp \mathrm{Lie}(\pi_1(\mathbb{P}^1 - \{0,1,\infty\})$ given by

$$\exp\big(\mathrm{Lie}[x,y,z|x+y+z=0]\big) \simeq \langle e^x, e^y, e^z | e^{x+y+z} = 1\rangle.$$

For Φ_{KZ}, the isomorphism is given by

$$e^X \mapsto e^{2i\pi x}$$

$$e^Y \mapsto \Phi_{KZ}(y,x)e^{2i\pi y}\Phi_{KZ}(x,y)$$
$$e^Z \mapsto e^{i\pi x}\Phi_{KZ}(z,x)e^{2i\pi z}\Phi_{KZ}(x,z)e^{-i\pi x}.$$

The product of the three terms is equal to 1 thanks to relation (II).

If instead if $\Phi \in GRT_0$, i.e. $\mu = 0$, it gives an automorphism of $\exp \mathrm{Lie}(\pi_1(\mathbb{P}^1 - \{0,1,\infty\})$ given by

$$e^x \mapsto e^x, \quad e^y \mapsto \Phi(y,x)e^y\Phi(x,y).$$

This gives rise to a group law on GRT_0 coming from the composition:

$$\Phi(x,y) \odot \Psi(x,y) = \Phi(x,y)\Psi(x, \Phi^{-1}y\Phi).$$

Each GRT_μ is a torsor under GRT_0 by composition.

Braid extension: Following Grothendieck, we can identify $\mathbb{P}^1 - \{0,1,\infty\}$ with the moduli space of genus zero curves with 4 marked points, and the free π_1 is then identified with the pure sphere braid on 4 strands.

Relation (III) ensures that the formality isomorphism of the 4-strand (free) group given by any associator above extends to one of the pro-unipotent pure sphere braid group on 5 strands, and elements of GRT_0 extend to automorphisms of the pro-unipotent pure 5-strand braid group.

Double shuffle relations satisfied by Φ_{KZ}

The two multiplications on mzvs can be expressed economically by two conditions on the power series.

• The first condition is

$$\Delta(\Phi_{KZ}) = \Phi_{KZ} \otimes \Phi_{KZ},$$

where $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\Delta(y) = y \otimes 1 + 1 \otimes y$. This means that Φ_{KZ} is *group-like*, i.e. it lies in $\exp(\mathrm{Lie}[x,y])$.

Viewed as a family of algebraic relations on the coefficients of Φ_{KZ}, it is exactly the shuffle relations on the multizetas.

• The second condition is obtained by considering the modified series

$$\Phi_{corr} = \exp\Big(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n}\zeta(y_n)y_1^n\Big)\pi_y(\Phi_{KZ})$$

where $\pi_y(\Phi_{KZ})$ is the projection of Φ_{KZ} onto the words ending in y. We rewrite the series Φ_{corr} in the variables $y_i = x^{i-1}y$, and the condition is

$$\Delta_*(\Phi_{corr}) = \Phi_{corr} \otimes \Phi_{corr},$$

where

$$\Delta_*(y_i) = \sum_{k+l=i} y_k \otimes y_l.$$

These relations are called the *double shuffle* relations on Φ_{KZ}.

Definition. Let DS_λ denote the set of power series in x, y with

- constant term 1
- no linear term
- coefficient λ for the monomial xy
- that satisfy the double shuffle relations.

Theorem. [Racinet, 2000] *The set DS_0 of double shuffle power series forms a group under the multiplication*

$$f(x,y) \odot g(x,y) = f(x,y)g(x, f^{-1}yf).$$

Theorem. [Furusho] *The group GRT_0 injects into DS_0 by the map*

$$\Phi(x,y) \mapsto \Phi(x,-y).$$

In other words, even if we don't know whether double shuffle relations imply *all* algebraic relations on multizeta values, we do know that they imply the associator relations.

Part II. A quick look at Écalle's mould theory

Very generally, a *mould* is a family $(P_r)_{r\geq 0}$ where P_r is a function of r commutative variables $u_1,\ldots,u_r$.

We will restrict attention to rational functions over $\mathbb{Q}$, so $P_0 \in \mathbb{Q}$ and $P_r \in \mathbb{Q}(u_1,\ldots,u_r)$ for $r \geq 1$. Let ARI denote the space of moulds with constant term 0 and $GARI$ the set of moulds with constant term 1.

The *swap* is an operation on moulds that is given by the following change of variables:

$$swap(A)(u_1,\ldots,u_r) = A(u_r, u_{r-1}-u_r,\ldots,u_1-u_2).$$

We can make $GARI$ into a non-commutative group by the multiplication law

$$(A\times B)(u_1,\ldots,u_r) = \sum_{i=0}^{r} A(u_1,\ldots,u_i)B(u_{i+1},\ldots,u_r).$$

Theorem. [easy] *Let $\mathcal{G}$ be the set of power series in x, y with constant term 1 in the kernel of the derivation $\partial_x(x) = 1$, $\partial_x(y) = 0$. There is an injective homomorphism*

$$\mathcal{G} \hookrightarrow GARI$$

given by rewriting elements in the non-commutative variables where $c_i = ad(x)^{i-1}(y)$ and linearly extending

$$c_{i_1}\cdots c_{i_r} \mapsto u_1^{i_1-1}\cdots u_r^{i_r-1}.$$

Note in particular that Écalle extended the usual notion of multiplication of non-commutative power series to **all moulds**.

The double shuffle relations satisfied by elements of DS translate onto the image moulds as follows.

A mould in $GARI$ is **symmetral** if for $1 \leq i \leq r-1$, we have

$$\sum_{wsh\big((u_1,\ldots,u_i),(u_{i+1},\ldots,u_r)\big)} P(w) = P(u_1,\ldots,u_i)P(u_{i+1},\ldots,u_r).$$

The first double shuffle relation on a power series $\Phi(x,y) \in \mathcal{G}$ is equivalent to the associated polynomial mould being symmetral.

A mould is **symmetril** if it satisfies a set of relations similar to symmtrality, but where the left-hand side is deduced from the stuffle relations by replacing every sum $u_i + u_j$ with the term

$$\frac{1}{u_i - u_j}\Big(A(\ldots,u_i,\ldots) - A(\ldots,u_j,\ldots)\Big).$$

For example the stuffle relation in depth 2 is given by

$$st\big((u_1),(u_2)\big) = (u_1,u_2) + (u_2,u_1) + (u_1+u_2),$$

and a mould A is *symmetril in depth 2* if it satisfies

$$A(u_1,u_2) + A(u_2,u_1) + \frac{1}{u_1-u_2}\big(A(u_1) - A(u_2)\big) = A(u_1)A(u_2).$$

The second double shuffle relation on power series Φ is equivalent to the swap of the corresponding polynomial mould being symmetril.

Recall that we had

Theorem. [Racinet, 2000] *The set DS_0 of double shuffle power series forms a group under the multiplication*

$$f(x,y) \odot g(x,y) = f(x,y)g(x, f^{-1}yf).$$

Racinet's proof is *difficult* and *non-intuitive.*

Just as Écalle defined a multiplication on $GARI$ that extends the ordinary multiplication of non-commutative power series to all moulds, he also defined a multiplication law $gari$ on $GARI$ that extends $\odot$ from polynomial moulds to all moulds.

We will show Écalle's remarkable untwisting theorem, which gives a proof of Racinet's theorem as an easy corollary.

Écalle's amazing untwisting map

Let $Ad_{gari}(P) \cdot A = gari(Q, A, P)$ where Q is the $gari$-inverse of P.

Theorem. [Écalle] *There exists a special mould $P \in GARI$ such that if a mould $A \in GARI$ is symmetral with symmetril swap (i.e. double shuffle), then $Ad_{gari}(P) \cdot A$ is symmetral with symmetral swap.*

The mould P is not polynomial, but it is closely related to the power series

$$\frac{ad(x)}{e^{ad(x)} - 1} \cdot y = \sum_{n \geq 0} \frac{B_n}{n!} ad(x)^n (y).$$

(Bernoulli numbers!)

• It is *much easier* to work with moulds that are symmetral with symmetral swap than symmetral with symmetril swap.

• It is *easy* to show that $GARI^{ev}_{symmetral/symmetral}$ is a group under the $gari$ multiplication (where ev denotes moulds that are even in depth 1).

• But by Écalle's Theorem, we have

$$GARI^{ev}_{symmetral/symmetril} = gari(P, GARI^{ev}_{symmetral/symmetral}, Q),$$

where $gari(P, Q) = 1$, so this is also a group under the $gari$ multiplication!

• Polynomial moulds form a subgroup, so $GARI^{pol,ev}_{symmetral/symmetril}$ is a subgroup.

• This group is exactly DS_0.

Part III. Multizeta values and associators in genus one

The starting point of genus one multizeta theory as developed by B. Enriquez is the pair of moulds

$$I^{A_\tau}(u_1,\ldots,u_r) = \int_{0<v_r<\cdots<v_1<1} F_\tau(u_1,v_1)\cdots F_\tau(u_r,v_r)\,dv_r\cdots dv_1$$

$$I^{B_\tau}(u_1,\ldots,u_r) = \int_{0<v_r<\cdots<v_1<\tau} F_\tau(u_1,v_1)\cdots F_\tau(u_r,v_r)\,dv_r\cdots dv_1,$$

with

$$F_\tau(u,v) = \frac{\theta(u+v;\tau)}{\theta(u;\tau)\theta(v;\tau)}$$

where θ is the (odd) Jacobi theta function and τ runs over the Poincaré upper half-plane.

Proposition. (easy) *Let A_τ be the mould defined by*

$$A_\tau(u_1,\ldots,u_r) = u_1\cdots u_r\, I^{A_\tau}(u_1,\ldots,u_r)$$

for each $r\geq 1$. Then A_τ is a group-like polynomial mould. The same holds for B_τ.

By a slight abuse of notation, we write A_τ and B_τ also for the power series in $\exp(\mathrm{Lie}[a,b])$ corresponding to these moulds under the mould map $c_{i_1}\cdots c_{i_r}\mapsto u_1^{i_1-1}\cdots u_r^{i_r-1}$ where $c_i = ad(a)^{i-1}(b)$.

Definition. The pair (A_τ,B_τ) is the *elliptic associator.* It was constructed as the elliptic analog of the Drinfel'd associator.

Just as the multizeta values are the coefficients of Φ_{KZ}, Enriquez defines the *elliptic analogs of multizeta values* to be the coefficients of the power series A_τ together with $2\pi i\tau$ (adding the coefficients of B_τ adds nothing). We write $\mathcal{EMZV}$ for the $\mathbb{Q}$-algebra generated by these elements.

The algebra of elliptic multizeta values

Let

$$Ber_x(y) = \frac{ad(x)}{e^{ad(x)} - 1} \cdot y = \sum_{n \geq 0} \frac{B_n}{n!} ad(x)^n(y).$$

Consider the elements

$$t_{01} = Ber_b(-a), \quad t_{02} = Ber_{-b}(a), \quad t_{01} = [a, b]$$

in $Lie[a, b]$, so $t_{01} + t_{02} + t_{12} = 0$. Set

$$A = \Phi_{KZ}(t_{01}, t_{12}) e^{2\pi i t_{01}} \Phi_{KZ}(t_{01}, t_{12})^{-1}$$

and

$$B = \Phi_{KZ}(t_{02}, t_{12}) e^{2\pi i t_{01}} \Phi_{KZ}(t_{01}, t_{12})^{-1}.$$

They are power series in exp(Lie$[a, b]$) with coefficients in $\mathcal{Z}[2\pi i]$.

Theorem. [Enriquez] *There exists an automorphism* g_τ *of*

$$\langle e^a, e^b \rangle \otimes \langle \text{iterated integrals of Eisenstein series } G_{2k} \rangle,$$

such that

$$A_\tau = g_\tau(A), \quad B_\tau = g_\tau(B).$$

Remark. For those who like $\mathbf{u}^{geom}$, the automorphism $g_\tau \in exp(\mathbf{u}^{geom})$.

Theorem. [Matthes-Lochak-S] *The algebra* $\mathcal{EMZV}$ *of elliptic multizeta values is isomorphic to the tensor product of* $\mathcal{Z}[2\pi i]$ *and the algebra generated by the coefficients of* g_τ*, i.e.*

$\mathcal{EMZV} \simeq \mathcal{Z}[2\pi i] \otimes_{\mathbb{Q}} \langle$certain subspace of iterated integrals of Eisenstein series$\rangle$.

Comparison of the genus zero and genus one associators

• As the Drinfel'd associator arises from the usual KZB equation, the pair (A_τ, B_τ) arise from an elliptic KZB equation based on the function F_τ;

• As Φ_{KZ} is the iterated integral of the KZB differential form $\frac{x}{v} + \frac{y}{1-v}\,dv$, A_τ is the iterated integral of the differential form $F_\tau(u,v)\,dv$;

• As the Drinfeld associator yields an isomorphism between

$$\pi_1^{pro-unip}(\mathbf{P}^1 - \{0,1,\infty\}) \simeq \langle e^X, e^Y, e^Z | e^X e^Y e^Z = 1\rangle$$

and the graded version

$$\exp(\mathrm{Lie}[x,y,z|x+y+z=0]) \simeq \langle e^x, e^y, e^z | e^{x+y+z} = 1\rangle,$$

so (A_τ, B_τ) gives an isomorphism between

$$\pi_1^{pro-unip}(T_1) \simeq \langle e^A, e^B, e^C | (e^A, e^B) e^C = 1\rangle$$

and the graded version

$$\exp(\mathrm{Lie}[a,b,c|[a,b]+c=0]) = \langle e^a, e^b, e^c | e^{[a,b]+c} = 1\rangle,$$

where T_1 is the once-punctured torus. The isomorphism is given by $e^A \mapsto A_\tau$, $e^B \mapsto B_\tau$.

• Viewing $\langle e^X, e^Y\rangle$ as the pro-unipotent π_1 of the thrice-punctured sphere identifies it with the pure sphere 4-strand braid group. The formality isomorphism on $\langle e^X, e^Y\rangle$ induced by Φ_{KZ} extends to one of the pure sphere 5-strand braid group [Kohno-Drinfeld]. Similarly, viewing $\langle e^A, e^B\rangle$ as the pro-unipotent π_1 of T_1 identifies it with the 1-strand torus braid group, and the formality isomorphism given by (A_τ, B_τ) extends to one one the the 2-strand torus braid group.

• The extension to the 5-strand braid group is ensured by the *associator relations* satisfied by Φ_{KZ}. Similarly, (A_τ, B_τ) satisfy *elliptic associator relations* arising from the fact that it induces a formality isomorphism of the 2-strand torus braid group.

Elliptic multizeta values

Technical restriction: From now on we work mod $2\pi i$ (keeping the same notation but considering all objects mod $2\pi i$). In particular since $A \equiv 1$ mod $2\pi i$, we replace it by $A^{1/2\pi i}$ mod $2\pi i$ (and the same for A_τ). We write $\overline{\mathcal{EMZV}}$ for the reduction of $\mathcal{EMZV}$ modulo the ideal generated by $2\pi i$.

Recall Écalle's special mould P. Set

$$\Psi = gari(Q, \Phi_{KZ}, P).$$

The mould Ψ is rational-valued, not polynomial. We fix this as follows.

For any mould P, let

$$\begin{aligned} dar(P)(u_1, \ldots, u_r) &= (u_1 \cdots u_r)\, P(u_1, \ldots, u_r) \\ dur(P)(u_1, \ldots, u_r) &= (u_1 + \cdots + u_r)\, P(u_1, \ldots, u_r). \end{aligned}$$

Set

$$\Delta^*(P) = 1 - dar\Big(Q^{-1} \times dur(Q)\Big)$$

where $gari(P, Q) = 1$ as above, i.e. Q is the $gari$-inverse of P.

Theorem. [Baumard-S] *The mould $E = \Delta^*(\Psi)$ is polynomial-valued.*

Because E is polynomial-valued, it comes from a power series in a and b which we also call E. Let $E_\tau = g_\tau(E)$.

Theorem. [S] *We have*

(i) there exists a unique automorphism of $\langle e^a, e^b \rangle$ such that $e^a \mapsto E$ and $e^{[a,b]}$ is fixed. This automorphism also satisfies

$$e^{t_{01}} \mapsto A, \quad e^b \mapsto B.$$

Composing with g_τ gives an automorphism such that

$$e^a \mapsto E_\tau, \quad e^{t_{01}} \mapsto A_\tau, \quad e^b \mapsto B_\tau.$$

(ii) The coefficients of E (with $2\pi i \tau$) give another system of generators of $\overline{\mathcal{EMZV}}$.

Definition. We call E the *elliptic generating series*, and the coefficients of E *elliptic multizeta values.*

What about elliptic double shuffle?

Definition. A group-like power series F satisfies the *elliptic double shuffle relations* if the associated mould is of the form

$$F = \Delta^*(G)$$

where G is symmetral with symmetral swap.

Theorem. *The elliptic generating series E_τ satisfies the elliptic double shuffle relations.*

Indeed, E satisfies them because

- $E = \Delta^*(\Psi)$
- $\Psi = gari(Q, \Phi_{KZ}, P)$
- By Écalle's untwisting theorem, since Φ_{KZ} is symmetral/symmetril, Ψ is symmetral/symmetral.

Finally, E_τ satisfies them because g_τ also does, which is an easy consequence of the construction of g_τ as an element of $exp(\mathbf{u}^{geom})$.

Consequence: If one prefers to have double shuffle type relations directly on A_τ, they can be obtained since the elliptic double shuffle relations on E_τ translate over to a double family of relations on A_τ: the group-like (shuffle) relations and a second family called the Fay relations studied by N. Matthes.

Final observation. Let **ds** be the double shuffle Lie algebra and **lds** the linearized double shuffle Lie algebra. Then symmetral/symmetril is the group-like version of the Lie algebra double shuffle relations and symmetral/symmetral is the group-like version of the linearized double shuffle relations (i.e. those of the associated graded of **ds** for the depth filtration). Thus the action of Écalle's mould P has the effect of changing the double symmetry to that of the associated graded. In particular the elliptic double shuffle relations are closer to the linearized double shuffle relations than to the double shuffle relations.