

**On mixed motivic sheaves
and weights for them**

Mikhail Bondarko, MPIM &
St. Petersburg State University
mbondarko@gmail.com

Weight structures and applications

Weight structures: "cousins" of t -structures (on a triangulated category \underline{C}); "two halves".

For $\underline{C} = K^b(\underline{B})$ or $= K(\underline{B})$:
 $\underline{C}_{w \leq 0}$ = complexes, homotopic to ones concentrated in degrees ≥ 0 ;
 $\underline{C}_{w \geq 0}$ = complexes, \sim in degrees ≤ 0 .

Weight structures give:

- weight complexes (vastly generalizing that of [GiS96]);
- \underline{C} -functorial weight filtrations and spectral sequences (generalising coniveau and Deligne's weight ones);
- virtual t -truncations (that cut (co)homology into "pure" pieces), adjacent t -structures;
- other new (co)homological functors;
- $K_i(\underline{C})$ for $i \leq 0$.

The heart $\underline{Hw} \subset \underline{C}$:

$\text{Obj} = \underline{C}_{w=0} = \underline{C}_{w \geq 0} \cap \underline{C}_{w \leq 0}$.

\underline{Hw} is additive; *negative*: $\forall M, N \in \underline{C}_{w=0}, i > 0 \implies \underline{C}(M, N[i]) = \{0\}$
($\underline{C}_{w=0} \perp \underline{C}_{w=i} = \underline{C}_{w=0}[i]$). Now,

Constructing weight structures is easy!

\forall negative $\underline{B} \subset \underline{C} \exists$ a *bounded* w on a subcategory of \underline{C} , with $\underline{B} \subset \underline{Hw}$.

Also some results for "big" \underline{C}
(compactly and well generated).

$S \in SH^c \subset SH$ gives $w_{sph}^c \subset w_{sph}$;
 $\underline{Hw} \cong$ (finitely generated) **free** abelian groups.

t_{sph} comes from cellular towers
& calculates singular homology.

Definitions

B is Kar-closed in \underline{C} if contains \underline{C} -retracts of its objects.

Definition 1 (B; D. Pauksztello). (i) $\underline{C}_{w \geq 0}, \underline{C}_{w \leq 0}$ are Karoubi-closed in \underline{C} .

(ii) $\underline{C}_{w \leq 0} \subset \underline{C}_{w \leq 0}[1], \underline{C}_{w \geq 0}[1] \subset \underline{C}_{w \geq 0}$.

(iii) **Orthogonality.** $\underline{C}_{w \leq 0} \perp \underline{C}_{w \geq 0}[1]$.

(iv) **Decompositions.** $\forall M \in \text{Obj } \underline{C}$
 $\exists A \rightarrow M \rightarrow B: A \in \underline{C}_{w \leq 0}, B \in \underline{C}_{w \geq 0}[1]$.

Similar to t -structures; yet different!

Shifted weight decompositions

$\forall M \in \text{Obj } \underline{C}, m \in \mathbb{Z}$:

$w_{\leq m}M \rightarrow M \rightarrow w_{\geq m+1}M \rightarrow (w_{\leq m}M)[1]$

with $w_{\leq m} \in \underline{C}_{w \leq m} = \underline{C}_{w \leq 0}[m]$ and

$w_{\geq m+1}M \in \underline{C}_{w \geq m+1} = \underline{C}_{w \geq 0}[m+1]$.

Only weakly functorial.

Two types of w for (co)motivic categories: Chow and Gersten ones.

Chow weight structures on motives

$w_{Chow}(\text{Spec } k)$ corresponding to $\text{Chow}(k) \subset DM_{gm}(k)$ (with $\mathbb{Z}[\frac{1}{p}]$ -linear coefficients).

Extends to $DM(k)$ and $DM_c(S) \subset DM(S)$.

This gives:

- A conservative(!) exact functor
 $t : DM_{gm} \rightarrow K^b(\text{Chow}); M \mapsto (M^i);$
- $\forall H : DM_{gm}^{op} \rightarrow \underline{A}$ (abelian category), $i \in \mathbb{Z}$, a subfunctor
 $W_i H : M \mapsto \text{Im}(H(w_{\geq i} M) \rightarrow H(M)).$
- Weight spectral sequence for a cohomological $H: E_1^{pq} = H^q(M^{-p}) \implies H^{p+q}(M); DM_{gm}$ -functorial starting from E_2 (which is determined by $t(M)$).
- $K_i(DM_{gm}) \cong K_i(\text{Chow})$ for $i \leq 0$.
- Pure functors, adjacent t_{Chow} on DM.

An application to conservativity

If $\text{char } k = 0$ then $\Omega'_{DR} = \Omega_{DR}^{t_{hom} \leq 0}$ is a (h.s.) ring object in (k -linear) $\text{DM}(k)$.

$$\begin{array}{ccccc}
 \text{Chow} & \longrightarrow & DM_{gm} & \xrightarrow{t} & K^b(\text{Chow}) \\
 \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 \\
 \text{Mot}_{alg} & \longrightarrow & \Omega'_{DR} - \text{Mod}_c & \xrightarrow{t'} & K^b(\text{Mot}_{alg}) \\
 & & \downarrow H_{DR}^* & & \\
 & & k - \text{vect}^{\mathbb{Z}} & &
 \end{array}$$

The first square from [BOg94, Corollary 7.4]; t and t' are weight complexes.

Since p_1 is conservative (by [Voe95]) and full, p_3 and p_2 are conservative.

If H_{DR}^* conservative (J. Ayoub), then $H_{DR}^* \circ p_2$ also is!

One can also apply weight spectral sequences here.

What if H_{DR}^* is conservative?

1) Then RH_{et} and vanishing cycles from $DM_c(S)$ are conservative for $S/\text{Spec } \mathbb{Q}$.

2) $w_{Chow}(k)$ is a \otimes -structure; hence t (along with H_{DR}^* !) are \otimes -functors.

Thus conservativity of $H_{DR}^* \implies$ "Nori motives" in DM_{gm} are Kimura-finite i.e. for an affine A/k and its generic hyperplane section Z both

$M_{gm}(Z \rightarrow A)$ and $t(M_{gm}(Z \rightarrow A))$ are finite-dimensional (either higher exterior or higher symmetric powers = 0).

Hence numerical motives are Kimura-finite (can one do better??!); motivic zetas are rational.

On relative motives and cross functors (Voevodsky, Ayoub, Cisinski, Deglise).

B = Noetherian separated excellent of finite Krull dimension;

B -schemes = separated f.t. $/B$,

\mathcal{D} : from B -schemes into (tensor) triangulated categories.

$\mathbf{1}_X$ = tensor unit of $\mathcal{D}(X)$ for a X/B .

For any B -morphism $f : X \rightarrow Y$
 adjoint $f^* : \mathcal{D}(Y) \rightleftarrows \mathcal{D}(X) : f_*$
 and $f_! : \mathcal{D}(X) \rightleftarrows \mathcal{D}(Y) : f^!$.

"Axioms": $f_* = f_!$ if f is proper; smooth and proper base change; gluing, etc.

For any $X/B : p_*(\mathbf{1}_{\mathbb{P}^1(X)}) \cong \mathbf{1}_X \oplus \mathbf{1}_X \langle -1 \rangle$;
 $-\langle n \rangle = - \otimes (\langle -1_X \rangle)^{-n} = -(n)[2n]$.

When w_{Chow} exists

The *Chow-negativity* axiom required:

$\mathbf{1}(X) \perp \mathbf{1}(X)\langle i \rangle[n] \quad \forall i \in \mathbb{Z}, n > 0, \forall$
regular X/B (or $X = \text{Spec } K$).

Examples

(i) $\mathcal{D}(-) = \text{DM}(-, R) =$ modules over
 $EM_R \in SH(-)$ (Voevodsky motives):

a) for "any" B and $R = \mathbb{Q}$;

b) for $1/p \in R$ and $\text{char } B = p$

(we set " $1/0 = 1$ ");

(ii) "Cobordism-modules" (setting b));

(iii) K -motives ($KH[\mathcal{S}^{-1}]$ -ones) $\forall B$:

$\mathcal{S} = \{\text{primes not invertible on } B\}$,

$\mathcal{DK}(X) = \text{KGl}'_X[\mathcal{S}^{-1}] - \text{Mod.}$

Properties of "weights"

Similar to [BBD82, §5.1] and [Hub97].

$\mathbf{1}_X \in \mathcal{D}(X)_{w_{Chow}=0}$ if X is regular;
 f^* and $f_!$ "respect $w_{Chow} \leq 0$ ";
 f_* and $f^!$ "respect $w_{Chow} \geq 0$ ".

$\mathcal{D}_{w_{Chow} \leq 0}(X)$ is "generated" by $f_!(\mathbf{1}_Y)\langle i \rangle$
for B -morphisms $f : Y \rightarrow X$, $i \in \mathbb{Z}$;

$\mathcal{D}_{w_{Chow} \geq 0}$ is "generated" by $f_*(\mathbf{1}_Y)\langle i \rangle$
for regular Y .

Hw contains/ generated by $f_*(\mathbf{1}_Y)\langle i \rangle$
for regular Y and proper f :

\mathcal{D} -Chow motives.

\mathcal{D} -Chow weight structures "glue";
can be detected pointwisely.

\forall regular proper R/U , open $j : U \rightarrow X$

$\text{Im}(K^*(R_X) \rightarrow K^*(R)) = W_0 H^*(j_!(u_! \mathbf{1}_R))$

\forall "proper regular model" R_X/X

and $H = \mathcal{DK}(X)(-, \mathbf{1}_X)$:

"integral part of K -theory" (a-la Scholl).

Transversality with (motivic) t -structures

$\forall t$ on \underline{C} gives: (i) abelian $\underline{Ht} \subset \underline{C}$
and (ii) t -homology $H : \underline{C} \rightarrow \underline{Ht}$.

$W^i H \forall M \in \underline{C}^{t=0}$: "just a filtration";
"better" if all w.s.s. for H E_2 -degenerate.

Transversality assumption:

$\forall M \in \underline{C}^{t=0}, i \in \mathbb{Z} \exists w_{\leq i} M \in \underline{C}^{t=0}$.

Examples: $MHM(-)$ (and so, graded polarizable Hodge complexes), 1-motives, Artin-Tate motives/ $\#$ -fields.

For motives implies: pure motives

$PM_i(S) = \text{Obj MM} \cap DM_{\mathbb{Q},c}(S)_{w_{Chow}=i}$
are semisimple (abelian); \exists Chow-Künneth

decompositions: $DM_c(S)_{w_{Chow}=0} = \bigoplus PM_i(S)[-i]$

"Standard" motivic conjectures (incl.

Murre's ones) over universal domains \implies

"nice weights" on $MM(S) \subset DM_{\mathbb{Q}}(S)$

\forall variety S (and more generally).

D.T.: $PM(S) = \bigoplus_{s \in S} j_{s!*} PM(s)[\dim(\bar{s})]$.

Thank you for your attention!

References

- [Ayo07] Ayoub J., *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (I)*, *Asterisque*, vol. 314, Soc. Math. France, 2007.
- [BBD82] Beilinson A., Bernstein J., Deligne P., *Faisceaux pervers*// *Astérisque* 100, 1982, 5–171.
- [BOg94] Bloch S., Ogus A., *Gersten’s conjecture and the homology of schemes*// *Ann. Sci. Éc. Norm. Sup.* v.7 (1994), 181–202.
- [Bon10] Bondarko M., *Weight structures vs. t -structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*// *J. of K-theory*, v. 6, i. 03, p. 387–504, 2010, see also <http://arxiv.org/abs/0704.4003>
- [Bon12] Bondarko M.V., *Weight structures and ‘weights’ on the hearts of t -structures*// *Homology, Homotopy and Appl.*, vol. 14(1), 2012, 239–261.
- [Bon14] Bondarko M.V., *Weights for relative motives: relation with mixed complexes of sheaves*// *Int. Math. Res. Notes*, vol. 2014, i. 17, 4715–4767; see also <http://arxiv.org/abs/1007.4543>.
- [Bon15] Bondarko M.V., *Mixed motivic sheaves (and weights for them) exist if ‘ordinary’ mixed motives do*// *Compositio Mathematica*, vol. 151, 2015, 917–956.
- [BoI15] Bondarko M.V., Ivanov M.A., *On Chow weight structures for cdh -motives with integral coefficients*// *Algebra i Analiz*, v. 27 (2015), no. 6, 14–40; see also <http://arxiv.org/abs/1506.00631>.
- [BoL16] Bondarko M.V., Luzgarev A.Ju., *On relative K -motives, weights for them, and negative K -groups*, preprint, <http://arxiv.org/abs/1605.08435>
- [CiD12] Cisinski D.-C., Déglise F., *Triangulated categories of mixed motives*, preprint, <http://arxiv.org/abs/0912.2110v3>.
- [CiD15] Cisinski D.-C., Déglise F., *Integral mixed motives in equal characteristic*// *Documenta Mathematica*, Extra Volume: Alexander S. Merkurjev’s Sixtieth Birthday, 145–194, 2015.

- [GiS96] Gillet H., Soulé C. *Descent, motives and K-theory*// J. f. die reine und ang. Math. v. 478, 1996, 127–176.
- [Heb11] Hébert D., *Structures de poids à la Bondarko sur les motifs de Beilinson*// Compositio Mathematica, vol. 147, is. 5, 2011, 1447–1462.
- [Hub97] Huber A., *Mixed perverse sheaves for schemes over number fields*, Compositio Mathematica, 108, 1997, 107–121.
- [Pau08] Pauksztello D., *Compact cochain objects in triangulated categories and co-t-structures*// Central European Journal of Mathematics, vol. 6, n. 1, 2008, 25–42.
- [Sch00] Scholl A., *Integral elements in K-theory and products of modular curves*, in: The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Science Series C, volume 548 (Kluwer, 2000), 467–489.
- [Voe95] Voevodsky V., *A nilpotence theorem for cycles algebraically equivalent to zero*// Int. Math. Res. Notices, 1995, 187–199.
- [Wil16] Wildeshaus J., *Notes on Artin-Tate motives*, in: *Panorama et Synthèses*, 2016, Autour des motifs, III, 49, 101–131